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CONSISTENCY PROOFS IN MODEL THEORY: A CONTRIBUTION TO *JENSENLEHRE**

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We prove the two-cardinal principle $(\kappa^{++}, \kappa) \to (\lambda^{++}, \lambda)$ consistent for all κ and λ, without the use of Jensen's morasses.

0. Introduction

We work in ZFC+GCH throughout.

We consider countable languages containing a designated singulary predicate U. If $\mathfrak{A} = (A, U^{\mathfrak{A}}, \ldots)$ is a structure for such a language, and κ' and κ are infinite cardinals, we say $\mathfrak{A}$ is of *type* (κ', κ) if card $A = \kappa'$ and card $U^{\mathfrak{A}} = \kappa$. The *two-cardinal transfer principle* $(\kappa', \kappa) \to (\lambda', \lambda)$ is the assertion that any countable first-order theory having a model of type (κ', κ) also has one of type (λ', λ). It of course suffices to consider theories formulated in a fixed countable language $\mathscr{L}$ containing U and "enough" other symbols (denumerably many n-ary predicates and function symbols for each n). By the *gap-two principle for* (κ, λ) we mean the principle $(\kappa^{++}, \kappa) \to (\lambda^{++}, \lambda)$.

Since there are only ω_1 $\mathscr{L}$-theories (first-order theories formulated in the language $\mathscr{L}$), clearly the gap-two principle holds for many pairs of cardinals. However, Silver [16] has shown that in ZFC+GCH we cannot prove the gap-two principle for any *particular* values of κ and λ, e.g. we cannot prove $(\omega_8, \omega_6) \to (\omega_5, \omega_3)$. Jensen (see [6], Chapters 13 and 14) has shown that if is a consequence of the Axiom of Constructibility ($V = L$), and hence consistent with ZFC+GCH, that the gap-two principle holds for all κ and λ. As Magidor says in [13], "These results due to Jensen $\cdots$ set a record in their technical subtlety". This is chiefly owing to the prominent role in Jensen's arguments played by the elaborate combinatorial structures called *morasses*.

Jensen in effect produces a fixed $\mathscr{L}$-theory T_0 and proves:

0.1. Theorem. *If an $\mathscr{L}$-theory T has a model of type (κ^{++}, κ) for some κ, then T is consistent with T_0.*

0.2. Theorem. *If T is an $\mathscr{L}$-theory consistent with T_0, and λ a regular cardinal such that there exists a λ-morass, then T has a model of type (λ^{++}, λ).*

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0.3. Theorem. *It is consistent with* ZFC+GCH *that* λ*-morasses exist for every* λ.

These theorems of Jensen clearly establish the gap-two principle for any κ and any regular λ. For singular λ the proof is unpublished and requires something worse than a morass (cf. [18]). Jensen derives the existence of morasses from $V=L$ using his fine structure theory (see [6], Chapter 13). He also has a proof (unpublished, but see [18]) of the consistency of morasses by a simpler, but still difficult, forcing construction.

We will show how to prove, by an almost purely model-theoretic argument, entirely avoiding morasses, the following result, which can be substituted for Jensen's 0.2 and 0.3 in proving the consistency of gap-two principles:

0.4. Theorem. *It is consistent with* ZFC+GCH *that the following should hold for all infinite cardinals* (*regular or singular*):
($^*\lambda$) *Every* $\mathscr{L}$*-theory consistent with* t_0 *has a model of type* (λ^{++}, λ).

We hope that by eliminating morasses from the proof we will make this area of model theory more palatable to those logicians who have found it thus far "distressingly like set theory".

Our inspiration in this work has been Jack Silver's high praise of Jensen's *results*, and his aesthetic criticisms of Jensen's *methods*. We are grateful to Professor Jensen for many patient explanations. Conversations with I. Juhasz, L.J. Stanley, several of our students at Madison, and participants in the Yale Logic Colloquium have contributed to the development of our ideas.

1. Review of Jensen's work

Our first task must be to produce the theory T_0 which is to figure in all our work.

1.1. Definition. Add to our fixed countable language $\mathscr{L}$ two new binary predicates E and $<$, thus forming the language $\mathscr{L}'$. Let T' be the $\mathscr{L}'$-theory containing the following axioms:

(i) $<$ is a linear order in which U forms an initial segment.

(ii) (Well-Foundedness) $\forall x_1 \cdots \forall x_n(\exists y\varphi(x_1 \cdots x_n, y) \rightarrow \exists y(\varphi(x_1 \cdots x_n, y)\ \&\ \forall z(z<y \rightarrow \neg\varphi(x_1 \cdots x_n, z))$, for every $\mathscr{L}'$-formula φ.

(iii) (Coding) $\forall x_1 \cdots \forall x_n(U(x_1)\ \&\ \cdots\ \&\ U(x_n) \rightarrow \exists y(U(y)\ \&\ \forall z(zEy \leftrightarrow (z=x_1 \vee \cdots \vee z=x_n))))$, for every $n>0$.

Add to $\mathscr{L}'$ a constant c, two singulary function symbols F and G, and a new symbol S^* for each symbol S of $\mathscr{L}'$, thus forming a language $\mathscr{L}^*$. S^* should be the same *kind* of symbol as S, e.g. a binary predicate if S is. For any $\mathscr{L}'$-formula φ, let φ^* be the result of replacing each symbol S by the corresponding S^*. Let T^* be the $\mathscr{L}^*$-theory extending T' and containing in addition the following axioms:

(iv) $\forall x_1 \cdots \forall x_n(\varphi(x_1 \cdots x_n) \to \varphi^*(F(x_1) \cdots F(x_n)))$, for every $\mathscr{L}'$-formula φ.

(v) $\forall x_1 \cdots \forall x_n(\varphi(x_1 \cdots x_n) \to \varphi^*(G(x_1) \cdots G(x_n)))$, for every $\mathscr{L}'$-formula φ.

(vi) $\forall x(U(x) \to x < c)$.

(vii) $\forall x(x < c \to F(x) = G(x))$.

(viii) $\forall x(F(x) <^* G(c))$.

(ix) $\forall x(U^*(x) \to \exists y(U(y) \,\&\, x = F(y))$.

Let T_1 be the set of all $\mathscr{L}$-sentences which are logical consequences of T^*. Thus any $\mathscr{L}$-theory T will be consistent with T^* if and only if it is consistent with T_1. Finally, we take as T_0 any primitive recursive $\mathscr{L}$-theory whose set of logical consequences is exactly T_1. Such a T_0 exists by a well-known trick of Craig's. Indeed from the list (i)–(ix) above of the axioms of T^* we could easily obtain an explicit algorithm for membership in a suitable T_0.

1.2. Proof of Theorem 0.1. The hard work for the proof of this theorem has already been done for us by Jensen (see [6, Chapter 14, middle p. 176–top p.181]). Since one of our purposes is to interest the reader who may have been put off by morasses in the model-theoretic aspects of Jensen's work, we make no apology for simply quoting some of his results. (Another approach, using ideas of Silver, may be found in [3, Chapter 6]).

By GCH, for any infinite cardinal μ, the set H_μ of sets of hereditary cardinality $<\mu$ has cardinality exactly μ. Let T be an $\mathscr{L}$-theory having a model $\mathfrak{M}$ of type (κ^{++}, κ) for some κ. We may suppose the universe of $\mathfrak{M}$ is $H_{\kappa^{++}}$ and that $U^{\mathfrak{M}} = H_\kappa$, since the cardinalities are right. Expand $\mathfrak{M}$ to an $\mathscr{L}'$-structure $\mathfrak{M}'$ by taking as the interpretation of $<$ a well-ordering of $H_{\kappa^{++}}$ in type κ^{++} in which H_κ forms an initial segment, and as the interpretation of E the membership relation $\in$ of set theory. Plainly $\mathfrak{M}' \models T'$.

Now let $\mathfrak{A} = (A, U^{\mathfrak{A}}, \ldots, <^{\mathfrak{A}}, E^{\mathfrak{A}})$ be countable, $\mathfrak{A} \equiv \mathfrak{M}'$, and suppose $\mathfrak{A}$ satisfies a suitable partial saturation condition. (Jensen uses what he calls "arithmetical U-saturation", but recursive saturation as in [2] would also suffice.) Then Jensen shows ([6, Chapter 14, p. 179, Lemma 7]) that there exist $e \in A$ and $\mathfrak{B} = (B, U^{\mathfrak{B}}, \ldots, <^{\mathfrak{B}}, E^{\mathfrak{B}})$ and h such that:

(i) $\mathfrak{A} \prec \mathfrak{B}$ ($\prec$ indicates elementary substructure).

(ii) $h: \mathfrak{A} \to \mathfrak{B}$ is an elementary embedding.

(iii) $a <^{\mathfrak{A}} e$, for all $a \in U^{\mathfrak{A}}$.

(iv) $a = h(a)$, for all $a <^{\mathfrak{A}} e$, in particular for all $a \in U^{\mathfrak{A}}$.

(v) $a <^{\mathfrak{B}} h(e)$, for all $a \in A$.

(vi) $U^{\mathfrak{B}} = U^{\mathfrak{A}}$.

Let $J: A \to B$ be an arbitrary bijection. Define an $\mathscr{L}^*$-structure $\mathfrak{C}$ as follows: The universe of $\mathfrak{C}$ is A, and $S^{\mathfrak{C}} = S^{\mathfrak{A}}$ for all $\mathscr{L}'$-symbols S. For, say, a binary predicate S of $\mathscr{L}'$, $(S^*)^{\mathfrak{C}} = \{(a_0, a_1): (j(a_0), j(a_1)) \in S^{\mathfrak{B}}\}$, and similary with other S in $\mathscr{L}'$. Let $c^{\mathfrak{C}} = e$, let $F^{\mathfrak{C}}$ be the restriction of j^{-1} to the subset A of B, and let $G^{\mathfrak{C}} = j^{-1}h$. It is not hard to see that $\mathfrak{C} \models T \cup T^*$, properties (i)–(vi) above of

Jensen's construction corresponding to axioms 1.1(iv)–(ix) of T^*. Thus T is consistent with T^*, and hence with T_0.

2. The gap-two principle for regular λ

Let T be an $\mathscr{L}$-theory consistent with T_0. As a first step towards proving Theorem 0.4, we will show in this section how to adjoin generically a model of T of type (ω_4, ω_2).

2.1. Definition. Let $\mathscr{L}'$, c be as in Definition 1.1 If $\mathfrak{A}$ is an $\mathscr{L}'$-structure, and e an element of the universe of $\mathfrak{A}$, we mean by $(\mathfrak{A}, e)$ the expansion of $\mathfrak{A}$ obtained by taking e as the interpretation of the constant c. We call e *praiseworthy* if $\mathrm{Th}(\mathfrak{A}, e)$, the first-order theory of this expanded structure, is consistent with T^*. (Note this can only happen if $\mathfrak{A}$ is already a model of T.) Note that if $\mathfrak{A}$ is *saturated* and $\mathrm{Th}\mathfrak{A}$ is consistent with T^*, then $\mathfrak{A}$ has an expansion to a model of T^*, hence certainly has a praiseworthy element.

2.2 Lemma. *Let $\mathfrak{A} = (A, U^{\mathfrak{A}}, \ldots)$ be a saturated $\mathscr{L}'$-structure, and $e \in A$. If e is praiseworthy, then there exist $\mathfrak{B}$ and h such that conditions 1.2*(i)–(vi) *hold. Moreover, $\mathfrak{B}$ may be taken saturated, and of the same cardinality as $\mathfrak{A}$.*

Proof. $(\mathfrak{A}, e)$ is saturated, and its theory consistent with T^*. Therefore it has an expansion to a saturated model $\mathfrak{A}^*$ of T^*. Let $\mathfrak{D}$ be the saturated $\mathscr{L}'$-structure with universe A and $S^{\mathfrak{D}} = (S^*)^{\mathfrak{A}*}$ for every $\mathscr{L}'$-symbol S. By axiom 1.1(iv) of $T^*, f = F^{\mathfrak{A}*}$ is an elementary embedding of $\mathfrak{A}$ into $\mathfrak{D}$, and hence an isomorphism between $\mathfrak{A}$ and an elementary substructure $\mathfrak{C}$ of $\mathfrak{D}$. Let $\mathfrak{B}$ be an elementary extension of $\mathfrak{A}$ and $j: \mathfrak{B} \to \mathfrak{D}$ an isomorphism making the following diagram commute:

$$\begin{tikzcd}
\mathfrak{B} \arrow[r, "j"] & \mathfrak{D} \\
\mathfrak{A} \arrow[u, hook] \arrow[r, "f"'] & \mathfrak{C} \arrow[u, hook]
\end{tikzcd}$$

Let $g = G^{\mathfrak{A}*}$ and $h = j^{-1}g$. Axioms 1.1(iv)–(ix) of T^* guarantee that e, $\mathfrak{B}$, and h satisfy conditions 1.2(i)–(vi). Plainly $\mathfrak{B}$ is saturated and of the same cardinality as $\mathfrak{A}$.

2.3. Definition. Since our $\mathscr{L}$-theory T is consistent with T_0, it is consistent with T^*, and so (by GCH) $T \cup T^*$ has a saturated model with universe ω_2. Let $\mathfrak{A}_T = (\omega_2, U_T, \ldots, <_T, E_T)$ be the $\mathscr{L}'$-reduct of this saturated model.

Let P_T be the set of all pairs consisting of (i) an $\mathscr{L}'$-structure $\mathfrak{A} = (A, U^{\mathfrak{A}}, \ldots, <^{\mathfrak{A}}, E^{\mathfrak{A}})$ with $\mathfrak{A} \equiv \mathfrak{A}_T$, $\mathfrak{A}$ saturated, $A \subseteq \omega_3$, card $A = \omega_2$, $\mathfrak{A}_T < \mathfrak{A}$, and $U^{\mathfrak{A}} = U_T$; together with (ii) a function Λ from a subset of $\omega_4 - \omega_3$ to the set of praiseworthy elements of $\mathfrak{A}$, which is order-preserving in the sense that if $\xi, \eta \in \operatorname{dom} A$, and $\xi < \eta$ (as ordinals), then $A(\xi) <^{\mathfrak{A}} A(\eta)$.

If $(\mathfrak{A}, A)$ and $(\mathfrak{A}', A')$ are elements of P_T, we say an elementary embedding h of $\mathfrak{A}'$ into $\mathfrak{A}$ is a *commendable* map from $(\mathfrak{A}', A')$ to $(\mathfrak{A}, A)$ if $h \restriction U_T$ is the identity, and A extends hA'. We say $(\mathfrak{A}, A) \lhd (\mathfrak{A}', A')$ if there exists a commendable map from the latter to the former, and $(\mathfrak{A}, A) \equiv (\mathfrak{A}', A')$ if there exist commendable maps in both directions. Clearly is reflexive (the identity map is commendable) and transitive (a composition of commendable maps is commendable), so $\equiv$ is an equivalence relation. Let us write $[\mathfrak{A}, A]$ for the equivalence class of $(\mathfrak{A}, A)$ and $\mathcal{P}_T$ for $\{[\mathfrak{A}, A]: (\mathfrak{A}, A) \in P_T\}$. Then $\lhd$ induces a partial order on $\mathcal{P}_T$. By abuse of language we call this partial order $\lhd$ as well, and write simply $\mathcal{P}_T$ for $(\mathcal{P}_T, \lhd)$. The partially ordered set $\mathcal{P}_T$ has maximal element $1_T = [\mathfrak{A}_T, \text{empty function}]$.

We now turn to proving some useful antichain and closure conditions for $\mathcal{P}_T$.

2.4. Definition. For $\alpha < \omega_2$, a sequence $\mathbf{S} = ((\mathfrak{A}_\beta, A_\beta)\colon \beta < \alpha) \in P_T^\alpha$ is *coherent* if for all $\gamma < \beta < \alpha$, $\mathfrak{A}_\gamma \subset \mathfrak{A}_\beta$ and the inclusion is a commendable map from $(\mathfrak{A}_\gamma, A_\gamma)$ to $(\mathfrak{A}_\beta, A_\beta)$. We define a function F_T on coherent sequences as follows: For $\mathbf{S}$ as above, let $\mathfrak{A} = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$, $A = \bigcup_{\beta<\alpha} A_\beta$. $\mathfrak{A}$ need not be saturated, but by a clever device of Chang [5] it will at least be U-saturated. (In order for Chang's device to be applicable, we need the Coding Axioms 1.1(iii). This is the only purpose of those axioms.) This means there is saturated $\mathfrak{B}$ (which we may take to have universe $\subset \omega_3$), of the same cardinality (ω_2) as $\mathfrak{A}$, elementarily extending $\mathfrak{A}$ (and hence $\equiv \mathfrak{A}_T$), with $U^{\mathfrak{B}} = U^{\mathfrak{A}} (= U_T)$. Pick such a $\mathfrak{B}$ and set $F(\mathbf{S}) = (\mathfrak{B}, A)$. Note $F_T(\mathbf{S}) \in P_T$, and for $\beta < \alpha$, the inclusion is a commendable map from $(\mathfrak{A}_\beta, A_\beta)$ to $F_T(\mathbf{S})$.

2.5 Lemma. *$\mathcal{P}_T$ is ω_2, ∞-distributive.*

Proof. We must show that if D_α, $\alpha < \omega_1$, are open dense subsets of $\mathcal{P}_T$, then so is their intersection. Openness is immediate. To prove density, consider an arbitrary $p_0 \in \mathcal{P}_T$. We define a coherent sequence $((\mathfrak{A}_\alpha, A_\alpha)\colon \alpha \leq \omega_1)$ as follows:

Pick $(\mathfrak{A}_0, A_0)$ such that $p = [\mathfrak{A}_0, A_0]$. Having $(\mathfrak{A}_\beta, A_\beta)$, let q be an element of D_β which is $\leq [\mathfrak{A}_\beta, A_\beta]$ (such exists by density). If $q = [\mathfrak{B}, A]$ and h is a commendable map from $(\mathfrak{A}_\beta, A_\beta)$ to $(\mathfrak{B}, A)$, we can easily construct an extension $\mathfrak{A}_{\beta+1}$ of $\mathfrak{A}_\beta$ with universe $\subset \omega_3$, and an isomorphism $g: \mathfrak{A}_{\beta+1} \to \mathfrak{B}$ with $g \restriction \mathfrak{A}_\beta = h$. Set $A_{\beta+1} = Ag$, so that the inclusion becomes a commendable map from $(\mathfrak{A}_\beta, A_\beta)$ to $(\mathfrak{A}_{\beta+1}, A_{\beta+1})$. At limits $\alpha \leq \omega_1$, set $(\mathfrak{A}_\alpha, A_\alpha) = F_T(((\mathfrak{A}_\beta, A_\beta)\colon \beta < \alpha))$. Then $p = [\mathfrak{A}_{\omega_1}, A_{\omega_1}]$ is $\leq p_0$ and belongs to $\bigcap_{\alpha<\omega_1} D_\alpha$, proving density.

2.6. Combinatorial Principle. Let μ be a regular cardinal, $\mathcal{F}$ a family of μ^+ subsets of μ^+, each of cardinality $< \mu$. Then there exist $\mathcal{F}' \subset \mathcal{F}$ of cardinality μ^+ and a fixed $X \subset \mu^+$ such that (i) X is an initial segment of every element of $\mathcal{F}'$, and (ii) for distinct $Y, Z \in \mathscr{F}'$, $Y-X$ and $Z-X$ are nonoverlapping (i.e., either every ordinal in $Y-X$ is less than every ordinal in $Z-X$, or vice versa).

Proof. Recall we are assuming GCH. Then 2.6 is a slight variant of [9, Lemma 6] which is itself a direct generalization of a result of Szpilrajn-Marczewski.

2.7. Lemma. *$\mathscr{P}_T$ has the ω_4-antichain property.*

Proof. Let $\mathscr{A} \subset \mathscr{P}_T$ have cardinality ω_4. We will find $p, q \in \mathscr{A}$ which are compatible in $\mathscr{P}_T$.

By GCH there are only ω_3 subsets A of ω_3 of cardinality ω_2. For each such A there are only ω_3 $\mathscr{L}'$-structures $\mathfrak{A}$ with universe A. For each such $\mathfrak{A}$ there can be at most $\omega_3 R \subset A$ such that R is the range of some function Λ with $(\mathfrak{A}, \Lambda) \in P_T$. Hence there exist $\mathscr{A}' \subset \mathscr{A}$ of cardinality ω_4 and fixed $\mathfrak{A} = (A, U_T, \ldots)$ and $R \subset A$, such that every $p \in \mathscr{A}'$ has form $p = [\mathfrak{A}, \Lambda_p]$ where range $\Lambda_p = R$. Note that since each Λ_p is order-preserving, dom Λ_p must be distinct for distinct $p \in \mathscr{A}'$.

Now apply 2.6 with $\mu = \omega_3$ and $\mathscr{F} = \{\text{dom } \Lambda_p : p \in \mathscr{A}'\}$. There must exist $p, q \in \mathscr{A}'$ and a common initial segment X of $Y = \text{dom } \Lambda_p$ and $Z = \text{dom } \Lambda_q$, such that every element of $Y-X$ is less than every element of $Z-X$. If for some ordinal ν, η is the νth element of Y, ζ the νth element of Z, and a the νth element of R (in the order $<^{\mathfrak{A}}$), then $\Lambda_p(\eta) = a = \Lambda_q(\zeta)$.

Let $e = \Lambda_p(\inf(Y-X))$. By Lemma 2.2 there exist a saturated $\mathfrak{B} = (B, U_T, \ldots)$ and a map h such that $e, \mathfrak{B}$, and h satisfy 1.2(i)–(vi). Define a map Λ from $Y \cup Z$ to B by:

$$\Lambda(\eta) = \Lambda_p(\eta) \quad \text{for} \quad \eta \in Y,$$

$$\Lambda(\zeta) = h(\Lambda_q(\zeta)) \quad \text{for} \quad \zeta \in Z.$$

Note that for $\xi \in Y \cap Z = X$, $\Lambda_p(\xi) = \Lambda_q(\xi) <^{\mathfrak{A}} e$, and h is the identity below e, so indeed $\Lambda_p(\xi) = h(\Lambda_q(\xi))$. Note also that if $\eta \in Y$ and $\zeta \in Z-X$, then $\eta < \zeta$ and $e \leqslant^{\mathfrak{A}} \Lambda_q(\zeta)$, so $\Lambda(\eta) = \Lambda_p(\eta) <^{\mathfrak{B}} h(e) \leqslant^{\mathfrak{B}} h(\Lambda_q(\zeta)) = \Lambda(\zeta)$. It is now not hard to see that Λ is order-preserving, that $(\mathfrak{B}, \Lambda) \in P_T$, that the inclusion of $\mathfrak{A}$ in $\mathfrak{B}$ and the map h are commendable maps of $(\mathfrak{A}, \Lambda_p)$ and $(\mathfrak{A}, \Lambda_q)$ respectively into $(\mathfrak{B}, \Lambda)$, and finally that $r = [\mathfrak{B}, \Lambda]$ is $\lhd$ both p and q, which are thus compatible.

2.8. Lemma. *For every $\xi \in \omega_4 - \omega_3$, $D_\xi = \{[\mathfrak{A}, \Lambda] \in \mathscr{P}_T : \xi \in \text{dom } \Lambda\}$ is dense in $\mathscr{P}_T$.*

Proof. Note that if $[\mathfrak{A}, \Lambda] = [\mathfrak{A}', \Lambda']$, then dom Λ = dom Λ'. Let $p = [\mathfrak{A}, \Lambda] \in \mathscr{P}_T$ be arbitrary, $\xi \in \omega_4 - \omega_3$, $\xi \notin \text{dom } \Lambda$.

Case 1: If there exists $\eta \in \text{dom } \Lambda$ with $\xi < \eta$, let η_0 be the least such, and set $e = \Lambda(\eta_0)$.

Case 2: Otherwise let e be any praiseworthy element of $\mathfrak{A}$.

In either case, apply Lemma 2.2 to obtain $\mathfrak{B} = (B, U_T, \ldots)$ and h satisfying 1.2(i)–(vi). Define a map Λ' with dom $\Lambda' = \text{dom } \Lambda \cup \{\xi\}$ as follows: In case 1, let

$A'(\eta)=A(\eta)$ for $\eta\in\operatorname{dom} A\cap\xi$, $A'(\xi)=e$, and $A'(\eta)=h(A(\eta))$ for $\eta\in\operatorname{dom} A-\xi$. In case 2, simply let $A'(\eta)=A(\eta)$ for $\eta\in\operatorname{dom} A$ and $A'(\xi)=h(e)$. in either case A' is order-preserving, $(\mathfrak{B}, A')\in P_T$, and $q=[\mathfrak{B}, A']$ is a member of D_ξ with $q\lhd p$. This shows D_ξ is dense.

2.9. Definition. A Boolean-valued extension $V^{\mathscr{B}}$ of the universe V of set theory is *laudable* if $V^{\mathscr{B}}$ has the same cardinals as V and the GCH still holds in $V^{\mathscr{B}}$. Let V_T be the extension of V determined by $\mathscr{P}_T$ (i.e. $V_T=V^{\mathscr{B}}$ where $\mathscr{B}$ is the complete Boolean algebra of regular open subsets of $\mathscr{P}_T$). Lemmas 2.5 and 2.7, standard forcing lemmas, and GCH tell us that V_T is laudable.

2.10. Lemma. *In V_T, T has a model of type (ω_4, ω_2).*

Proof. We work in V_T. Let G ba a $\mathscr{P}_T$-generic set. Elements of G are highly compatible: If $p_1\cdots p_n\in G$, then there is a $q\in G$ which is $\lhd$ all p_i. Combining this fact with Lemma 2.8, we see that for any $\xi_1\cdots\xi_n\in\omega_4-\omega_3$, there is a $q=[\mathfrak{A}, A]\in G$ with $\xi_1\cdots\xi_n\in\operatorname{dom} A$.

Add to $\mathscr{L}'$ new constants $c(u)$ for $u\in U_T$ and $c(\xi)$ for $\xi\in\omega_4-\omega_2$, thus forming a large language $\mathscr{L}^+$. If $\varphi(x_1\cdots x_m, y_1\cdots y_n)$ is an $\mathscr{L}'$-formula and $(\mathfrak{A}, A)\in P_T$, we say $(\mathfrak{A}, A)$ *certifies* the $\mathscr{L}^+$-sentence $\varphi(c(\xi_1)\cdots c(\xi_m),\ c(u_1)\cdots c(u_n))$ if $\xi_1\cdots\xi_m\in\operatorname{dom} A$ and $\mathfrak{A}\models\varphi(A(\xi_1)\cdots A(\xi_m),\ u_1\cdots u_n)$. Note that if $(\mathfrak{A}, A)\lhd(\mathfrak{A}', A')$, then any sentence certified by $(\mathfrak{A}', A')$ is also certified by $(\mathfrak{A}, A)$.

It is now not hard to see that $W=\{\psi:\psi$ is an $\mathscr{L}^+$-sentence & $(\exists[\mathfrak{A}, A]\in G)((\mathfrak{A}, A)$ certifies $\psi)\}$ is a consistent, complete $\mathscr{L}^+$-theory extending $T\cup T'$. Moreover this theory contains $c(u)\neq c(v)$, $c(\xi)\neq c(\eta)$, and $c(u)\neq c(\xi)$ for distinct $u, v\in U_T$ and distinct $\xi,\eta\in\omega_4-\omega_2$. Thus we may choose $\mathfrak{M}\models W$ with $c(u)^{\mathfrak{M}}=u$ and $c(\xi)^{\mathfrak{M}}=\xi$ for all $u\in U_T$ and $\xi\in\omega_4-\omega_2$.

By the Well-Foundedness Axioms 1.1(ii) of T', The substructure $\mathfrak{N}$ of $\mathfrak{M}$ whose universe N consists of all *definable* elements of $\mathfrak{M}$ forms an *elementary* substructure of $\mathfrak{M}$. So $\mathfrak{N}\models T$, and clearly card $N=\omega_4$. We will show $U^{\mathfrak{N}}=U_T$.

For let $a\in U^{\mathfrak{N}}$. Since a is definable, there exist an $\mathscr{L}'$-formula φ and $\xi_1\cdots\xi_m\in\omega_4-\omega_2$ and $u_1\cdots u_n\in U_T$ such that

$$\mathfrak{M}\models\exists! z(U(z)\ \&\ \varphi(\xi_1\cdots\xi_m, u_1\cdots u_n, z))\ \&\ \varphi(\xi_1\cdots\xi_m, u_1\cdots u_n, a).$$

There must be $[\mathfrak{A}\ A]\in G$ such that $(\mathfrak{A}, A)$ certifies

$$\exists! z(U(z)\ \&\ \varphi(c(\xi_1)\cdots c(\xi_m), c(u_1)\cdots c(u_n), z).$$

There must be $v\in U^{\mathfrak{A}}=U_T$ such that

$$\mathfrak{A}\models\varphi(A(\xi_1)\cdots A(\xi_m), u_1\cdots u_n, v).$$

But now we can only have $a=v\in U_T$. Thus indeed $U^{\mathfrak{N}}=U_T$, which has cardinality ω_2, and T has a model $\mathfrak{N}$ of type (ω_4, ω_2).

2.11. The gap-two principle for $\lambda = \omega_2$.

Let I be the set of all $\mathscr{L}$-theories consistent with T_0. For each $T \in I$ construct a partial order $\mathscr{P}_T$ as above. Let $\mathscr{P}(\omega_2)$ be the product $\Pi_{T \in I}\mathscr{P}_T$, $V(\omega_2)$ the corresponding extension of the universe V. Though distributivity is not always preserved by products, our proof of Lemma 2.5 (using the function F_T) readily extends to show that $\mathscr{P}(\omega_2)$ is ω_2, ∞-distributive. Hence there are no new subsets of ω_2 in $V(\omega_2)$ beyond those already present in V, and (hence) all cardinals $\leqslant \omega_3$ are preserved in $V(\omega_2)$, and in $V(\omega_2)$ $2^\kappa = \kappa^+$ for $\kappa \leqslant \omega_2$. Being a product of ω_1 partial orders each having the ω_4-antichain property, $\mathscr{P}(\omega_2)$ also has this property. Hence all cardinals $\geqslant \omega_4$ are preserved in $V(\omega_2)$, and in $V(\omega_2)$ $2^\kappa = \kappa^+$ for $\kappa \geqslant \omega_3$. Thus $V(\omega_2)$ is laudable. (The standard forcing lemmas we have been using may be found in [3] or [8].) Every $T \in I$ has a model of type (ω_4, ω_2) in $V(\omega_2)$. Moreover (since there are no new subsets of ω), no new $\mathscr{L}$-theories appear in $V(\omega_2)$ beyond those already present in V. Finally, being consistent with T_0 is obviously an absolute property of $\mathscr{L}$-theories. Hence inside $V(\omega_2)$ the following is true for $\lambda = \omega_2$:

$(*\lambda)$ Every $\mathscr{L}$-theory consistent with T_0 has a model of type (λ^{++}, λ).

The construction of partial orders $\mathscr{P}(\lambda)$ to give laudable extensions in which $(*\lambda)$ holds is straightforward for other regular uncountable λ. The construction for $\lambda = \omega$ requires special comment. First of all, saturated models are not available in this case, but fortunately the properties of recursively saturated countable models (see [2]) are in such good analogy with the properties of saturated models that we can use them instead with scarcely any change in our arguments. Second, we cannot take as $\mathscr{P}(\omega)$ simply the product of partial orders constructed for each $T \in I$, for we would lose the ω_2 antichain property. Instead we must use only those elements of the product which have the form $(p_T : T \in I)$ with p_T the trivial (maximal) element of the relevant partial order for all but countably many $T \in I$. With these two adjustments the construction proceeds for $\lambda = \omega$ much as for uncountable regular λ.

§3. The gap-two principle for singular λ, and for all λ all once

Again let T be a fixed $\mathscr{L}$-theory consistent with T_0. Let $\rho = \omega_{\omega_1}$. We show how to adjoin generically a model of T of type (ρ^{++}, ρ). Since saturated models do not in general exist in cardinality ρ, we need to use something else. Ideas of Jensen and Silver from [17] do the job.

3.1. Definition. Let $\mathfrak{A} = (A, \ldots)$ be a structure of cardinality ρ. A *ranking* of $\mathfrak{A}$ is a map r from A to the set of successor ordinals $< \omega_1$, such that for any such ordinal i, $A_i^r = \{a \in A : r(a) \leqslant i\}$ is the universe of an elementary substructure $\mathfrak{A}_i^r$ of $\mathfrak{A}$ of cardinality ω_i. r(a) is called the *r-rank* of a, and the pair $(\mathfrak{A}, r)$ is called a *ranked* structure. The ranking r and the ranked structure $(\mathfrak{A}, r)$ are *(U-)special* if each $\mathfrak{A}_i^r$ is (U-)saturated. The structure $\mathfrak{A}$ is *(U-)special* if it admits a (U-) special ranking.

If $(\mathfrak{A}, r)$, $(\mathfrak{B}, s)$ are ranked structures, $i < \omega_1$, and $h : \mathfrak{A} \to \mathfrak{B}$ and elementary embedding, then h is an *i-map* of $(\mathfrak{A}, r)$ into $(\mathfrak{B}, s)$ if $\max(i, r(a)) = \max(i, s(h(a)))$ for all $a \in A$.

The notion of *i-isomorphism* is similarly defined. $(\mathfrak{A}, r) <_i (\mathfrak{B}, s)$ if $\mathfrak{A} < \mathfrak{B}$ and the inclusion is an i-map of $(\mathfrak{A}, r)$ into $(\mathfrak{B}, s)$. $(\mathfrak{A}, r) =_i (\mathfrak{A}, s)$ is similarly defined. Note that if $i < j$, then any i-map is a fortiori a j-map. We call an elementary embedding $H : \mathfrak{A} \to \mathfrak{B}$ an ω_1-map of the ranked structure $(\mathfrak{A}, r)$ into the ranked structure $(\mathfrak{B}, s)$ if it is an i-map for some $i < \omega_1$.

Note that if $\mathfrak{A} = (A, U^{\mathfrak{A}}, \ldots)$ is, say, an $\mathcal{L}'$-structure of cardinality $\mathcal{R}$, and r a special ranking of $\mathfrak{A}$, and $e \in A$, then r will not in general be a ranking of the expansion $(\mathfrak{A}, e)$. It will only be a ranking if $r(e) = 1$, and in this case it remains a special ranking of the expansion. If $r(e) > 1$, we can only say that there exists *another* special ranking r' of $\mathfrak{A}$ with $(\mathfrak{A}, r) =_{r(e)} (\mathfrak{A}, r)')$ and $r'(e) = 1$, and that this r' is a special ranking of $(\mathfrak{A}, e)$. Apart from this small point, the properties of special ranked structures are close analogues (indeed, immediate corollaries) of the familiar properties of saturated structures.

Bearing all this in mind, the reader should have no great difficulty in verifying the following analogue of Lemma 2.2:

3.2. Lemma. *Let $(\mathfrak{A}, r)$ be a special ranked $\mathcal{L}'$-structure of cardinality ρ, e a praiseworthy element of $\mathfrak{A}$ (i.e.* $\mathrm{Th}(\mathfrak{A}, e)$ *is consistent with T^*), $i = r(e)$. Then there exist a special ranked structure $(\mathfrak{B}, s)$ and a map h such that:*

(i) $(\mathfrak{A}, r) <_i (\mathfrak{B}, s)$
(ii) *h is an i-map from $(\mathfrak{A}, r)$ to $(\mathfrak{B}, s)$*
(iii) –(vi) *of* 1.2 *hold.*

3.3. Definition. Fix a special $\mathcal{L}'$-structure $\mathfrak{A}_T = (\rho, U_T, \ldots)$ with $\mathfrak{A}_T \models T$ and $\mathrm{Th}\mathfrak{A}_T$ consistent with T^*. Fix a special ranking r_T of $\mathfrak{A}_T$. Let P_T be the set of all triples consisting of (i) an $\mathcal{L}'$-structure $\mathfrak{A} = (A, U^{\mathfrak{A}}, \ldots)$ of cardinality ρ, (ii) a ranking r of $\mathfrak{A}$, and (iii) a map Λ, such that $(\mathfrak{A}, r)$ is special, $A \subset \rho^+$, $(\mathfrak{A}_T, r_T) <_1 (\mathfrak{A}, r)$, $U^{\mathfrak{A}} = U_T$, and Λ maps a subset of $\rho^{++} - \rho^+$ order-preservingly into the set of praiseworthy elements of $\mathfrak{A}$.

If $(\mathfrak{A}, r, \Lambda)$, $(\mathfrak{A}', r', \Lambda') \in P_T$ and $h : \mathfrak{A}' \to \mathfrak{A}$ is an elementary embedding, we say h is a *commendable* map from the latter triple to the former if h is an ω_1-map, $h \mid U_T$ is the identity, and Λ extends $h\Lambda'$. Proceeding as in Section 2 we can now define $\lhd$, $[\mathfrak{A}, r, \Lambda]$, $\mathcal{P}_T$, etc.

A crucial lemma is the analogue of Lemma 2.5.

3.4. Lemma. *$\mathcal{P}_T$ is ρ^+, ∞-distributive.*

Proof. Call $\mathbf{S} = ((\mathfrak{A}_\beta, r_\beta, A_\beta) : \beta < \alpha) \in P_T^\alpha$, α a limit $\leq \rho$, *coherent* if

(i) for $\gamma < \beta < \alpha$, $\mathfrak{A}_\gamma \subset \mathfrak{A}_\beta$ and the inclusion is a commendable map from $(\mathfrak{A}_\gamma, r_\gamma, A_\gamma)$ to $(\mathfrak{A}_\beta, r_\beta, A_\beta)$;

(ii) for limits β, γ with $\gamma < \beta < \alpha$, $(\mathfrak{A}_\gamma, r_\gamma) <_1 (\mathfrak{A}_\beta, r_\beta)$;

(iii) for $1 \leq \nu < \omega_1$ and limits β with $\omega_\nu \leq \beta < \alpha$, $\{a \in A_\beta : r_\beta(a) \leq \nu\} \subset A_{\omega_\nu}$.

We define a function F_T on coherent sequences as follows: For $\mathbf{S}$ as above, let $\mathfrak{A} = (A, U_T, \ldots)$ be $\bigcup_{\beta<\alpha} \mathfrak{A}_\beta$, $A = \bigcup_{\beta<\alpha} A_\beta$. We distinguish three cases.

Case 1: $\alpha = \beta + \omega$ for some limit $\beta < \alpha$.
In this case, fix a strictly increasing function $I : \omega \to \omega_1$ such that $I(0) = 1$, $I(1) > \nu$, where card $\beta = \omega_\nu$, and such that $(\mathfrak{A}_{\beta+n}, r_{\beta+n}) <_{I(n+1)} (\mathfrak{A}_{\beta+n+1}, r_{\beta+n+1})$ for all n. For successor ordinals μ with $I(n) \leq \mu < I(n+1)$, let $\mathfrak{B}_\mu$ be the elementary substructure of $\mathfrak{A}$ with universe $B_\mu = \{a \in A_{\beta+n} : e_{\beta+n}(a) \leq \nu\}$. For $\mu >$ sup range I, let $\mathfrak{B}_\mu$ have universe $B_\mu = \bigcup_{n<\omega} \{a \in A_{\beta+n} : r_{\beta+n}(a) \leq \nu\}$. It is not hard to see that the $\mathfrak{B}_\mu$ form an elementary chain whose union is $\mathfrak{A}$, and that card $B_\mu = \omega_\mu$. For $\mu <$ sup range I, $\mathfrak{B}_\mu$ is saturated. For larger μ it is at least U-saturated. Thus setting $r(a) =$ (the least μ with $a \in B_\mu$) gives a U-special ranking of $\mathfrak{A}$.

It is now not hard to see that there exists a $\mathfrak{C} = (C, U_T, \ldots)$ and a special ranking s of $\mathfrak{C}$ such that $(\mathfrak{A}, r) <_1 (\mathfrak{C}, s)$ and $\mathfrak{C}_\mu^s = \mathfrak{A}_\mu^r$ for $\mu <$ sup range I. Set $F_T(\mathbf{S}) = (\mathfrak{C}, s, A)$.

Case 2: α is a limit of limits, and $\alpha < \rho$.

For successor ordinals $\mu < \omega_1$ let $\mathfrak{B}_\mu$ be the elementary substructure of $\mathfrak{A}$ with universe $B_\mu = \bigcup_{\beta<\alpha, \beta \text{ a limit}} \{a \in A_\beta : r_\beta(a) \leq \mu\}$. Using clause (ii) in the definition of coherence it is not hard to see that the $\mathfrak{B}_\mu$ from an elementary chain whose union is $\mathfrak{A}$. If $1 \leq \nu < \omega_1$ and card $\alpha = \omega_\nu$, then for $\mu \leq \nu$, $B_\mu = \{a \in A_{\omega_\mu} : r_{\omega_\mu}(a) \leq \mu\}$, and $\mathfrak{B}_\mu$ is a saturated model of cardinality ω_μ. For $\mu > \nu$, $\mathfrak{B}_\mu$ is at least U-saturated. Thus $r(a) =$ (the least μ with $a \in B_\mu$) defines a U-special ranking of $\mathfrak{A}$.

It is now not hard to see that there exists a $\mathfrak{C} = (C, U_T, \ldots)$ and a special ranking s of $\mathfrak{C}$ such that $(\mathfrak{A}, r) <_1 (\mathfrak{C}, s)$ and $\mathfrak{C}_\mu^s = \mathfrak{A}_\mu^r$ for $\mu \leq \nu$. Set $F_T(\mathbf{S}) = (\mathfrak{C}, s, A)$

Case 3: $\alpha = \rho$.

For successor ordinals $\mu < \omega_1$, let $\mathfrak{B}_\mu$ be the elementary substructure of $\mathfrak{A}$ with universe $B_\mu = \{a \in A_{\omega_\mu} : r_{\omega_\mu}(a) \leq \mu\}$. It is not hard to see that $r(a) =$ (the least μ with $a \in B_\mu$) defines a special ranking of $\mathfrak{A}$. Set $F_T(\mathbf{S}) = (\mathfrak{A}, r, A)$.

The proof can now be completed along the same lines as Lemma 2.5.

3.5. The two-cardinal principle for $\lambda = \rho$, for other singular λ, and for all λ at once (Proof of Theorem 0.4).

We leave to the reader the rest of the proof that $\mathcal{P}_T$ produces a laudable extension in which T has a model of type (ρ^{++}, ρ). We also leave to the reader the construction of a partial order $\mathcal{P}(\rho)$ producing a laudable extension in which the following is true for $\lambda = \rho$:

$(*\lambda)$ Every $\mathcal{L}$-theory consistent with T_0 has a model of type (λ^{++}, λ).

The construction of a similar $\mathscr{P}(\lambda)$ for other singular λ presents no additional difficulties.

Now let $\mathscr{P}(\infty)$ consist of all sequences $p=(p_\lambda : \lambda<\mu_p, \lambda$ a cardinal), where μ_p is a cardinal depending on p, each $p_\lambda \in \mathscr{P}(\lambda)$, the partial order constructed above, and for inaccessible $\mu \leqslant \mu_p$, card$\{\lambda<\mu : p_\lambda \neq$ the trivial (maximal) element of $\mathscr{P}(\lambda)\}<\mu$. Partially order $\mathscr{P}(\infty)$ by setting $p \leqslant q$ if and only if $\mu_p \geqslant \mu_q$ and for all $\lambda<\mu_q, p_\lambda \leqslant q_\lambda$ in $\mathscr{P}(\lambda)$. Then it is not hard to see that $\mathscr{P}(\infty)$ produces a laudable extension in which $(*\lambda)$ is true for all λ, proving Theorem 0.4.

By the methods of [12] it can be shown that $\mathscr{P}(\infty)$ preserves measurability of cardinals. Certain other large cardinal properties, e.g. Mahlo, are also preserved. Strong compactness seems to be lost.

3.6. Further consequences of $(*\lambda 1$. Assume $(*\lambda)$. Then the following hold:

(i) $(\kappa^{++}, \kappa) \rightarrow (\lambda^{++}, \lambda)$ for all κ.

(ii) The set of $\mathscr{L}$-sentences true in all structures of type (λ^{++}, λ) is RE.

(iii) If T is an $\mathscr{L}$-theory, and every finite subset of T has a model of type (λ^{++}, λ), then so does T itself.

(iv) If φ is an $\mathscr{L}$-sentence, and it is consistent with ZFC+GCH that for some κ, φ has a model of type (κ^{++}, κ), then φ in fact has a model of type (λ^{++}, λ).

Proof. (i) is immediate from $(*\lambda)$ and Theorem 0.1. In (ii), RE means "recursively enumerable in any reasonable Gödel numbering of the language". The set referred to in (ii) is, by $(*\lambda)$ and Theorem 0.1, just the set of logical consequences of the primitive recursive set of axioms T_0, which is of course RE. (iii) follows, since a theory has a model of type (λ^{++}, λ) if and only if it is consistent with T_0.

To see (iv), the most interesting of the lot, note that if $M \models$ZFC+GCH+$(\exists\kappa)(\varphi$ has model of type $(\kappa^{++}, \kappa))$, then since $M \models$ Theorem 0.1, $M \models (\varphi$ is consistent with $T_0)$. This, however, is a Π^0_1 statement about φ, and is absolute. So φ really is consistent with T_0, and so has a model of type (λ^{++}, λ) by $(*\lambda)$.

A great many combinatorial principles follow from $(*\lambda)$. For example, Silver [16] shows that the existence of a λ^+-Kurepa tree is equivalent to the existence of a model of type (λ^{++}, λ) for a certain $\mathscr{L}$-formula. Since the existence of Kurepa trees is consistent with ZFC+GCH, by 3.6(iv) $(*\lambda)$ implies the existence of a λ^+-Kurepa tree. Another example is Silver's $W(\lambda^+)$ for which see [3] or [11] which has innumerable tropological and combinatorial consequences (among them all the principles of [14] and [15]).

3.7. Extensions of our results.

The syntax of the logic $\mathscr{L}[Q_1, Q_2]$ is like that of ordinary first-order logic except for the presence of two generalized quantifiers Q_1 and Q_2 which behave syntactically just like $\forall$ and $\exists$. In the λ-semantics, $Q_1 x\varphi(x)$ is taken to mean that there exist at least $\lambda^+ x$ such that $\varphi(x)$, and $Q_2 x\varphi(x)$ that there exist $\lambda^{++} x$ such that $\varphi(x)$. By coding devices going back to Fuhrken [7] and Vaught [18], 3.6(ii) and (iii) yield axiomatizability and compactness results for $\mathscr{L}[Q_1, Q_2]$.

Our methods can be extended to deal with slightly larger languages. We could, for instance, allow countably infinite conjunctions and disjunctions. In the case $\lambda = \omega$, we could also allow the Magidor–Maltiz quantifier $Q_1^2 xy\varphi(x, y)$, interpreted in the ω-semantics to mean that there exists an uncountable set X such that $\varphi(x, y)$ holds for all $x, y \in X$. But for these extensions the analogue of Theorem 0.1 cannot simply be quoted from Jensen's work, and we will not, therefore, enter into these generalizations here.

Several years ago Jensen proved the consistency of—indeed, derived from $V = L$—the gap-three principle $(\kappa^{+++}, \kappa) \to (\lambda^{+++}, \lambda)$ for all κ and λ, along with the similar gap-n principle for all n. This work has never been published, and the versions of it circulating in manuscript are somewhat incomplete, but already show that the gap-three proof leaves the "record in technical subtlety" set by the gap-two proof far behind. It would therefore be especially desirable to extend the methods of this paper to cover the gap-three principle. The chief obstacle is the absence of a suitable analogue of the combinatorial result 2.6.

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